GRAPH OF LINEAR TRANSFORMATIONS OVER $\mathbb{R}$

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Abstract. In this paper, we study a connection between graph theory and linear transformations of finite dimensional vector spaces over $\mathbb{R}$ (the set of all real numbers). Let R^m, R^n be finite vector spaces over R, and let L be the set of all non-trivial linear transformations from R^m into R^n. An equivalence relation $\sim$ is defined on L such that two elements $f, k \in L$ are equivalent, $f \sim k$, if and only if $\ker(f) = \ker(k)$. Let $m, n \geq 1$ be positive integers and $V_{m,n}$ be the set of all equivalence classes of $\sim$. We define a new graph, $G_{m,n}$, to be the undirected graph with vertex set equals to $V_{m,n}$, such that two vertices, $[x], [y] \in V_{m,n}$ are adjacent if and only if $\ker(x) \cap \ker(y) \neq 0$. The relationship between the connectivity of the graph $G_{m,n}$ and the values of m and n has been investigated. We determine the values of m and n so that $G_{m,n}$ is a complete graph. Also, we determine the diameter and the girth of $G_{m,n}$.

1. Introduction

Let R be a commutative ring with $1 \neq 0$. Recently, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures), as well as, studying the interplay between ring-theoretic and graph-theoretic properties; see the survey articles [11], [10], [38] and [45]. In particular, as in [17], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [28], who let all the elements of R be vertices and was mainly interested in coloring. The zero-divisor graph of a ring R has been studied extensively by many authors, for example see([2]-[9], [12], [21]-[22], [37]-[43], [46]-[53], [57]). David. F. Anderson and the first-named author [13] introduced the *total graph* of R, denoted by $T(\Gamma(R))$. We recall from [13] that the total graph of a commutative ring R is the (simple) graph $\Gamma(R)$ with vertices R, and distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph (as in [13]) has been investigated in [8], [7], [6], [5], [45], [47], [51], [34] and [55]; and several variants of the total graph have been studied in [4], [14], [15], [16], [20], [27], [33], [30], [31], [32], [35], [36], and [44].

Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. In 2014, A. Badawi [26] introduced the annihilator graph of R. We recall from [26] that the annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. See the survey article [23]. It follows that each edge (path) of the classical zero-divisor of R is an edge (path) of $AG(R)$. For further investigations of $AG(R)$,

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see [19], [50], and [56]. In 2015, A. Badawi, investigated the *total dot product graph* of R [25]. In this case $R = A \times A \times \cdots \times A$ (n times), where A is a commutative ring with nonzero identity, and $1 \leq n < \infty$ is an integer. The *total dot product graph* of R is the (undirected) graph denoted by $TD(R)$, with vertices $R^* = R \setminus \{(0, 0, \ldots 0)\}$. Two distinct vertices are adjacent if and only if $x \cdot y = 0 \in A$, where $x \cdot y$ denote the normal dot product of x and y. The *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, ..., 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. In [25], both graphs $TD(R)$ and $ZD(R)$ are studied. The total dot product graph was recently further investigated in [1].

Other types of graphs attached to groups and rings were studied (for example) in [6], [8],[27], [37], [39]–[43], and [44].

Let G be a graph. Two vertices v_1, v_2 of G are said to be *adjacent* in G if v_1, v_2 are connected by an edge of G and we write $v_1 - v_2$. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the *diameter* of G is $\text{diam}(G) = \sup\{ d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles).

We say G is *connected* if there is a path in G from u to v for every $u, v \in V$. Therefore, a graph is said to be *disconnected*, if there exist at least two vertices $u, v \in V$ that are not joined by a path. We say that G is *totally disconnected* if no two vertices of G are adjacent. We denote the complete graph on n vertices by K_n, recall that a graph G is called complete if every two vertices of G are adjacent.

In this paper, we introduce a connection between graph theory and linear transformations of finite dimensional vector spaces over $\mathbb{R}$ (the ring of all real numbers). Let U and W be finite dimensional vector spaces over $\mathbb{R}$, such that $m = dim(U)$ and $n = dim(W)$. Since every finite dimensional vector space over $\mathbb{R}$ with dimension k is isomorphic to $\mathbb{R}^k$, we conclude that U is isomorphic to $\mathbb{R}^m$ and W is isomorphic to $\mathbb{R}^n$. Let $m, n \geq 1$ be positive integers and $L = \{t : \mathbb{R}^m \to \mathbb{R}^n \mid t$ is a nontrivial linear transformation from $\mathbb{R}^m$ into $\mathbb{R}^n\}$. If $s, t \in L$, then we say that s is equivalent to t, and we write $s \sim t$ if and only if $Ker(s) = Ker(t)$. Clearly, $\sim$ is an equivalence relation on L. For each $t \in L$, the set $[t] = \{s \in L | s \sim t\}$ is called the *equivalence class* of t. Let $V_{m,n}$ be the set of all equivalence classes of $\sim$. For positive integers $m, n \geq 1$, let $G_{m,n}$ be a simple undirected graph with vertex set $V_{m,n}$ such that two distinct vertices $[f], [k] \in V_{m,n}$ are adjacent if and only if $Ker(f) \cap Ker(k) \neq \{(0, \cdots, 0)\} \subset \mathbb{R}^m$.

2. Results

Remark 2.1. *If a graph G has one vertex, then we say that G is totally disconnected. Note that some authors state that such graph is connected.*

We have the following result.

Theorem 2.2. *The undirected graph $G_{m,1}$ is totally disconnected if and only if $m = 1$ or $m = 2$. Furthermore, if $m = 1$, then $V_{1,1} = \{[t]\}$ for some $t \in L$.*

Proof. Assume $m = 1$. Let $[t] \in V_{1,1}$. Since $t \in L$ (i.e., t is a nontrivial linear transformation from $\mathbb{R}$ into $\mathbb{R}$), we conclude that $dim(Range(t)) = 1$. Since $dim(Ker(t)) + dim(Range(t)) = m = 1$ and $dim(Range(t)) = 1$, we conclude that

$Ker(t) = \{0\}$. Thus $f \in [t]$ for every $f \in L$. Hence $V_{1,1} = \{[t]\}$ for some $t \in L$. Thus $G_{1,1}$ is totally disconnected by Remark 2.1.

Assume $m = 2$. Let $[t], [f] \in V_{2,1}$ be two distinct vertices. Since $t, f \in L$ (i.e., t, f are nontrivial linear transformations from $\mathbb{R}^2$ into $\mathbb{R}$), we conclude that $dim(Range(t)) = dim(Range(t)) = 1$. Since $dim(Ker(t)) + dim(Range(t)) = m = 2$ and $dim(Range(t)) = 1$, we conclude that $dim(Ker(t)) = 1$. Similarly, $dim(Ker(f)) = 1$. Since $t, f \in L$, and $dim(Ker(t)) = dim(Ker(f)) = 1$, we conclude that $Ker(t)$ and $Ker(f)$ are distinct lines passing through the origin $(0, 0)$. Thus $Ker(t) \cap Ker(f) = \{(0, 0)\}$. Hence $[t], [f]$ are nonadjacent. Thus $G_{2,1}$ is totally disconnected.

Now assume $m > 2$. We show that $G_{m,1}$ is connected. Let, $[t], [w] \in V_{m,1}$ be two distinct vertices. We show that $\ker(f) \cap \ker(k) \neq \{(0, \cdots, 0)\}$ for some $f \in [t]$ and $k \in [w]$. Let $\mathbf{M}_f$ be the standard $1 \times m$ matrix representation of f for some $f \in [t] \in V_{m,1}$ and $\mathbf{M}_k$ be the standard $1 \times m$ matrix representation of k for some $k \in [w] \in V_{m,1}$. By hypothesis, $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$. Say, $\mathbf{M}_f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \end{bmatrix}$ and $\mathbf{M}_k = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1m} \end{bmatrix}$

Let, $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$ and consider the system, $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ k_{11} & k_{12} & \cdots & k_{1m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since, $m > 2$, the number of equations $<$ the number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Therefore, $\ker(f) \cap \ker(k) \neq \mathbf{0}$, that is, the vertices $[t]$ and $[w]$ are adjacent. Further, since $[t]$, $[w]$ were chosen randomly, we conclude that the graph $G_{m,1}$ is complete for $m > 2$.

□

Theorem 2.3. *For $m = 1$ or $m = 2$, the undirected graph $G_{2,n}$ is totally disconnected for every positive integer $n \geq 1$.*

Proof. Assume $m = 1$ and $n \geq 1$ be a positive integer. Then by the proof of Theorem 2.2, we conclude that $V_{1,n} = \{[t]\}$ for some $t \in L$. Hence $V_{1,n}$ is totally disconnected by Remark 2.1.

Assume $m = 2$, and let $[t], [w] \in V$ be two distinct vertices. We want to show $\ker(f) \cap \ker(k) = 0$ for some $f \in [t]$ and $k \in [w]$. We may assume that neither $Ker(f) = 0$ nor $Ker(k) = 0$. Hence $dim(Ker(f)) = dim(Ker(k)) = 1$. Thus $Ker(f) \cap Ker(k) = \{(0, 0)\}$. Since $[f]$, $[k]$ were chosen randomly, we conclude that the graph $G_{2,n}$ is totally disconnected for $m = 2$. □

Theorem 2.4. *The graph $G_{m,n}$ is complete if and only if $m \geq 2n + 1$.*

Proof. Let $[t], [w] \in V$ such that $Ker(f) \neq 0$ and $Ker(k) \neq 0$ for some $f \in [t]$ and $k \in [w]$. Let $\mathbf{M}_f$ be the standard $n \times m$ matrix representation of $[f]$, $\mathbf{M}_k$ be the standard $n \times m$ matrix representation of $[k]$, and let $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$

Assume, $(x_1, x_2, \cdots, x_m) \in \mathbf{R}^m$ is a solution to $\mathbf{M}_{fk}\mathbf{x} = 0$, that is,

$$\begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}_{2n\times m} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n\times 1}$$

Let $r = \text{rank}\,(\mathbf{M}_{fk})$.

Assume $m \geq 2n+1$. We show $\ker(f) \cap \ker(k) \neq 0$. Since $r \leq 2n$ and $m \geq 2n+1$, we have number of equations $<$ number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = 0$ has infinitely many solutions, or $\text{null}\,(\mathbf{M}_{fk}) \neq 0$. Therefore, $\ker(f) \cap \ker(k) \neq 0$, that is the vertices $[t]$ and $[w]$ are adjacent. Since $[t]$ and $[w]$ are chosen randomly, we conclude that the graph $G_{m,n}$ is complete for $m \geq 2n+1$.

Conversaly, assume that $G_{m,n}$ is complete. We show that $m \geq 2n+1$. Suppose that $m < 2n+1$. We show that $G_{m,n}$ is not complete. Let $[t], [w] \in V$ such that $Ker(f) \neq 0$ and $Ker(k) \neq 0$ for some $f \in [t]$ and $k \in [w]$.

Case I: Suppose $r = m$.
We conclude that $\mathbf{M}_{fk}$ has m independent rows, say $R_1, R_2, \cdots, R_m$.
Consider the system,

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}^T$ is an invertible $m \times m$ matrix, we have $\text{null}\big(\begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}\big)^T = (0, 0, \cdots, 0)$. Thus $\ker(t) \cap \ker(w) = 0$. Hence the vertices $[t]$ and $[w]$ are nonadjacent

Case II: Suppose $r < m$. Thus we have the following system:

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since number of equations $<$ number of unknown variables, we conclude that $\text{null}\left(\begin{bmatrix} R_1 & R_2 & \cdots & R_r \end{bmatrix}^T\right) \neq (0, 0, \cdots, 0)$. This implies $\ker(f) \cap \ker(k) \neq 0$. Hence the vertices $[t]$ and $[w]$ are adjacent.

Since the vertices $[t]$ and $[w]$ can either be adjacent or nonadjacent, we conclude that the graph $G_{m,n}$ is not complete for every $1 \leq m < 2n+1$. □

Theorem 2.5. *Consider the undirected graph $G_{m,n}$. Assume $m \leq n$ and $m \neq 1$ or $m \neq 2$. Then $G_{m,n}$ is connected and $diam(G_{m,n}) = 2$.*

Proof. Let $[t]$, $[w] \in V$ such that $[t]$ and $[w]$ are nonadjacent. Choose $f \in [t]$ and $k \in [w]$. Then $\text{rank}\,(M_f) \neq m$ and $\text{rank}\,(M_k) \neq m$, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.

Assume $\operatorname{rank}(M_f) = m - i$, where $i \in \mathbf{N}$, $i \neq 1$, and $\operatorname{rank}(M_k) = m - j$, where $j \in \mathbf{N}$, $j \neq 1$. Then choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [h] \in V_{m,n}$, such that $[t]-[h]-[w]$.

Assume that $\operatorname{rank}(M_f) = m - 1$ and $\operatorname{rank}(M_k) = m - 1$. Then M_f has $m - 1$ independent rows, $R_1, R_2, \ldots, R_{m-1}$. Since $[t]$ and $[w]$ are nonadjacent, M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbf{R}^m$. Let $K \neq R$ be a non-zero row in M_k. Hence $K \in \operatorname{rowspace}(M_k)$. Since $K \in \mathbf{R}^m$, we have:

$$K = c_1R_1 + c_2R_2 + \cdots + c_{m-1}R_{m-1} + c_mR$$

Let $Y = K - c_mR$. Thus $Y \in \operatorname{rowspace}(M_k)$, (since both K and c_mR are $\in \operatorname{rowspace}(M_k)$), and $Y \in \operatorname{rowspace}(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some $d \in [h] \in V_{m,n}$. Since $Y \in \operatorname{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f. Thus $\operatorname{null}(M_{fd}) \neq 0$, since $\operatorname{rank}(M_{fd}) = m-1$. Hence $\ker(f) \cap \ker(d) \neq 0$. Hence $[t]$, $[h]$ are connected by an edge. Similarly, since $Y \in \operatorname{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Thus $\operatorname{null}(M_{kd}) \neq 0$, since $\operatorname{rank}(M_{kd}) = m - 1$. Hence $\ker(d) \cap \ker(k) \neq 0$. Thus $[h]$ and $[w]$ are adjacent. Therefore, we have $[t] - [h] - [w]$.

□

Example 2.6. *Suppose $m = 3$ and $n = 4$. So we are considering the graph $G\left([t] : \mathbf{R}^3 \to \mathbf{R}^4\right)$, where $m \leq n$, and $m \neq 1$ or $m \neq 2$, as given in Theorem 2.5. Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = 0_{m=3}$), and $[T] \neq 0, [L] \neq 0$. Let $f \in [T]$, and $k \in [L]$. Since $[T]$ and $[L]$ are non-trivial vertices, then $\operatorname{rank}(M_f) \neq m$ and $\operatorname{rank}(M_k) \neq m$, where M_f and M_k are the standard matrix representations of f and k.*
Suppose,

$$M_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3}, M_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3}$$

Let $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{8\times3}$
It can be easily seen that $\operatorname{rank}(M_{fk}) = 3$, which implies that $\operatorname{null}(M_{fk}) = 0$. Therefore, $\ker(f) \cap \ker(k) = 0$, that is the vertices $[T]$ and $[L]$ are not adjacent. We have:
$\operatorname{rank}(M_f) = 2 = 3 - 1 = m - 1$, *and* $\operatorname{rank}(M_k) = 2 = 3 - 1 = m - 1$.

Then M_f has 2 independent rows R_1 and R_2, such that $R_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$. The vertices $[T]$ and $[L]$ are not adjacent, thus M_k has one row R, such that $\{R_1, R_2, R\}$ are independent and form a basis for $\mathbf{R}^m$, where $m = 3$. In this example, $R = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Let $K \neq R$ be a non-zero row in M_k, $K = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$. $K \in \text{rowspace}(M_k)$ and since $K \in \mathbf{R}^3$ it can be written as a linear combination of $\{R_1, R_2, R\}$ as follows:

$$K = 1.R_1 + 1.R_2 - R = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Let $Y = K - (-1).R = K + R = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

This implies $Y \in \text{rowspace}(M_k)$ and $Y \in \text{rowspace}(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3}$*, be the standard matrix representation of some $d \in [W]$.*

Since $Y \in \text{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f. Thus $\text{null}(M_{fd}) \neq 0$ since $\text{rank}(M_{fd}) = 2$. Hence $\ker(T) \cap \ker(W) \neq 0$. Hence $[T], [W]$ are adjacent. Similarly, since $Y \in \text{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Hence $\text{null}(M_{kd}) \neq 0$ since $\text{rank}(M_{kd}) = 2$. Thus $\ker(L) \cap \ker(W) \neq 0$. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$.

Theorem 2.7. *Consider the undirected graph $G_{m,n}$. Assume that $n < m \leq 2n$ and $m \neq 1$ or $m \neq 2$. Then $G_{m,n}$ is connected and diam$(G_{m,n}) = 2$.*

Proof. Let $[T]$, $[L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = 0_m$), and $[T] \neq 0$, $[L] \neq 0$. Let, $f \in [T]$ and $k \in [L]$, then $\text{rank}(M_f) < m$ and $\text{rank}(M_k) < m$, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.

Assume that $n + 1 < m \leq 2n$. Then $\text{rank}(M_f) = n - i$, where $i = 0, 1, 2, \ldots$, and $\text{rank}(M_k) = n - j$, where $j = 0, 1, 2, \ldots$. Thus we can choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.

Assume that $m = n + 1$. Then we have three cases. **Case I**. Assume that $\text{rank}(M_f) = n = m - 1$, and $\text{rank}(M_k) = n - j$, where $j = 1, 2, \ldots$. Then we can choose any non-zero row, say Y from M_f, (Note that M_f is the matrix with the higher rank), to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.
Case II. Assume that $\operatorname{rank}(M_f) = n - i$, where $i = 1, 2, \ldots$ and $\operatorname{rank}(M_k) = n - j$, where $j = 1, 2, \ldots$. In this case any non-zero row Y can be chosen either from M_f or M_k, to form M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

. is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.
Case III. Assume that $\operatorname{rank}(M_f) = n$ and $\operatorname{rank}(M_k) = n$. Then M_f has n independent rows $R_1, R_2, \ldots, R_n$. Since $[T]$ and $[L]$ are not adjacent, M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbf{R}^m = \mathbf{R}^{n+1}$. Let $K \neq R$ be a non-zero row in M_k. Hence $K \in \operatorname{rowspace}(M_k)$. Since $K \in \mathbf{R}^{n+1}$, we have:

$$K = c_1R_1 + c_2R_2 + \cdots + c_nR_n + c_{n+1}R$$

Let $Y = K - c_{n+1}R$. Hence $Y \in \operatorname{rowspace}(M_k)$, (since both K, $c_{n+1}R \in \operatorname{rowspace}(M_k)$), and $Y \in \operatorname{rowspace}(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some $d \in [W]$.

Since $Y \in \operatorname{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f, $\operatorname{null}(M_{fd}) \neq 0$ since $\operatorname{rank}(M_{fd}) = n$. Hence $\ker(T) \cap \ker(W) \neq 0$. Thus $[T], [W]$ are adjacent. Similarly, since $Y \in \operatorname{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Hence $\operatorname{null}(M_{kd}) \neq 0$ since $\operatorname{rank}(M_{kd}) = n$. Thus $\ker(L) \cap \ker(W) \neq 0$. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$. □

Example 2.8. *Suppose $m = 4$ and $n = 3$ and consider the graph $G_{4,3}$. Note that $n < m \leq 2n$, $m \neq 1, 2$ and and $m = n+1$. Thus m, n satisfy the given hypothesis in Theorem 2.7. Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent. Let $f \in [T]$, and $k \in [L]$. Then* $\operatorname{rank}(M_f) < m$ *and* $\operatorname{rank}(M_k) < m$*, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m = 3 \times 4$. Suppose,*

$$M_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4}, M_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 4}$$

Let $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{6\times 4}$. It can be easily seen that $\operatorname{rank}(M_{fk}) = 4$*, which implies that* $\operatorname{null}(M_{fk}) = 0$*. Therefore,* $\ker(f) \cap \ker(k) = 0$*, that is, the vertices $[T]$ and*

$[L]$ are not adjacent. Hence $\text{rank}(M_f) = 3 = n$*, and* $\text{rank}(M_k) = 3 = n$*. Then M_f has 3 independent rows R_1, R_2, and R_3, such that* $R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $R_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$*, and* $R_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$*. The vertices $[T]$ and $[L]$ are not adjacent, thus M_k has one row,* $R = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$*, such that $\{R_1, R_2, R_3, R\}$ is an independent set which forms a basis for $\mathbf{R}^4$. Let $K \neq R$ be a non-zero row in M_k,* $K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$*. Since* $K \in \text{rowspace}(M_k)$ *and $K \in \mathbf{R}^4$, it can be written as a linear combination of $\{R_1, R_2, R_3, R\}$ as follows:*

$$K = 0.R_1 + 1.R_2 + 0.R_3 + (-1).R = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Let, $Y = K - (-1).R = K + R = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$.

This implies $Y \in \text{rowspace}(M_k)$ *and* $Y \in \text{rowspace}(M_f)$*. Let,* $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix}_{3\times 4} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$*, be the standard matrix representation of some $d \in [W]$.*

Since $Y \in \text{rowspace}(M_f)$*, Y becomes a zero row through row operations using the rows in M_f. Thus* $\text{null}(M_{fd}) \neq 0$*, since* $\text{rank}(M_{fd}) = 3$*. Hence* $\ker(T) \cap \ker(W) \neq 0$*. Thus $[T], [W]$ are adjacent. Similarly, since* $Y \in \text{rowspace}(M_k)$*, Y becomes a zero row through row operations using the rows in M_k. Thus* $\text{null}(M_{kd}) \neq 0$ *since* $\text{rank}(M_{kd}) = 3$*. Hence* $\ker(L) \cap \ker(W) \neq 0$*. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$.*

Theorem 2.9. *Assume that $G_{m,n}$ is connected. Then $gr(G_{m,n})) = 3$.*

Proof. $[T], [L] \in V$, such that $[T]$ and $[L]$ are adjacent, $\ker(T) \cap \ker(L) \neq 0$ and $[T] \neq 0, [L] \neq 0$. Let, $f \in [T]$ and $k \in [L]$, then M_f and M_k are the standard matrix representations of f and k with size $n \times m$. Suppose, that each matrix M_f and M_k, is composed of only one row, R_f and R_k that are independent of each other since f and k are in different equivalence classes $[T]$ and $[L]$. M_f and M_k can be written as follows:

$$M_f = \begin{bmatrix} R_f \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}, M_k = \begin{bmatrix} R_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$$

Let $Y = R_f + R_k$. Since Y is a linear combination of two linearly independent rows, then the set $\{Y, R_f, R_k\}$ is also linearly independent.

Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some non-trivial linear transformation d. Since Y is independent of both R_f and R_k, M_d is not row-equivalent to either M_f or M_k, hence d is in a different equivalence class from both f and k, say $d \in [W]$. Since $\ker(T) \cap \ker(L) \neq 0$, we have $\text{null}(M_{fk}) \neq 0$, which implies $\text{null}(M_{fd}) \neq 0$ and $\text{null}(M_{kd}) \neq 0$. Therefore, we have, $[T] - [L] - [W] - [T]$. This forms the shortest possible cycle. Hence $gr(G_{m,n})) = 3$. □

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References

[1] Abdulla, M., Badawi, A., On the dot product graph of a commutative ring II, 25 Int. Electron. J. Algebra **28**, 61–175 (2020).

[2] Akbari, S., Maimani, H. R., Yassemi, S.: When a zero-divisor graph is planar or a complete r-partite graph. J Algebra. **270**, 169–180 (2003).

[3] Akbari, S., Mohammadian, A., On the zero-divisor graph of a commutative ring. J Algebra. **274**, 847-855 (2004).

[4] Abbasi, A., Habib, S.: The total graph of a commutative ring with respect to proper ideals. J. Korean Math. Soc. **49**, 85–98 (2012)

[5] Akbari, S., Heydari, F.: The regular graph of a non-commutative ring. Bulletin of the Australian Mathematical Society (2013) Doi: 10.1017/S0004972712001177

[6] Akbari, S., Aryapoor, M., Jamaali, M.: Chromatic number and clique number of subgraphs of regular graph of matrix algebras. Linear Algebra Appl. **436**, 2419–2424 (2012).

[7] Akbari, S., Jamaali, M., Seyed Fakhari, S.A.: The clique numbers of regular graphs of matrix algebras are finite. Linear Algebra Appl. **43**, 1715—1718 (2009).

[8] Akbari, S., Kiani, D., Mohammadi, F., Moradi, S.: The total graph and regular graph of a commutative ring. J. Pure Appl. Algebra **213**, 2224-2228 (2009).

[9] Anderson, D. D., Naseer, M., Beck's coloring of a commutative ring. J Algebra. **159**, 500-514 (1993).

[10] Anderson, D.F., Axtell, M., Stickles, J.: Zero-divisor graphs in commutative rings. In : Fontana, M., Kabbaj, S.E., Olberding, B., Swanson, I. (eds.) Commutative Algebra Noetherian and Non-Noetherian Perspectives, pp. 23-45. Springer-Verlag, New York (2010).

[11] Anderson, D. F., Badawi, A.: "The Zero-Divisor Graph of a Commutative Semigroup: A Survey, DOI: 10.1007/978-3-319-51718-6_2." In Groups, Modules, and Model Theory Surveys and Recent Developments, edited by Manfred Droste, László Fuchs, Brendan Goldsmith, Lutz Strüngmann, 23-39. Germany/NewYork: Springer, 2017.

[12] Anderson, D.F., Badawi, A.: On the zero-divisor graph of a ring. Comm. Algebra **36**, 3073-3092 (2008).

[13] Anderson, D.F., Badawi, A.: The total graph of a commutative ring. J. Algebra **320**, 2706-2719 (2008).

[14] Anderson, D.F., Badawi, A.: The total graph of a commutative ring without the zero element. J. Algebra Appl. (2012) doi: 10.1142/S0219498812500740.

[15] Anderson, D.F., Badawi, A.: The generalized total graph of a commutative ring. J. Algebra Appl. (2013) doi: 10.1142/S021949881250212X.

[16] Anderson, D.F., Fasteen, J., LaGrange, J.D.: The subgroup graph of a group. Arab. J. Math. **1**, 17-27 (2012).

[17] Anderson. D.F., Livingston, P.S.: The zero-divisor graph of a commutative ring. J. Algebra **217**, 434-447 (1999). 434-447.

[18] Anderson, D.F., Mulay, S.B.: On the diameter and girth of a zero-divisor graph. J. Pure Appl. Algebra **210**, 543-550 (2007).

[19] Afkhami, M., Khashyarmanesh, K., and Sakhdari, S. M.: The annihilator graph of a commutative semigroup, J. Algebra Appl. 14, (2015) [14 pages] DOI: 10.1142/S0219498815500152

[20] Atani, S.E., Habibi, S.: The total torsion element graph of a module over a commutative ring. An. Stiint. Univ.'Ovidius Constanta Ser. Mat. **19**, 23-34 (2011).

[21] Axtel, M, Coykendall, J. and Stickles, J. : Zero-divisor graphs of polynomials and power series over commutative rings, Comm.Algebra **33**, 2043-2050 (2005).

[22] Axtel, M., Stickles, J.: Zero-divisor graphs of idealizations. J. Pure Appl. Algebra **204**, 235-243 (2006).

[23] Badawi, A.: Recent results on annihilator graph of a commutative ring: A survey. In Nearrings, Nearfields, and Related Topics, edited by K. Prasad et al, (11 pages), New Jersey: World Scientific, 2017.

[24] Badawi, A.: On the Total Graph of a Ring and Its Related Graphs: A Survey. In Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and

Polynomial Functions, DOI 10.1007/978-1-4939-0925-4 3, edited by M. Fontana et al. (eds.), 39-54. New York: Springer Science, 2014.
[25] Badawi, A., *On the dot product graph of a commutative ring*, Comm. Algebra **43**, 43-50 (2015).
[26] Badawi, A.: On the annihilator graph of a commutative ring, Comm. Algebra, Vol.(42)(1), 108-121 (2014), DOI: 10.1080/00927872.2012.707262.
[27] Barati, Z., Khashyarmanesh, K., Mohammadi, F., Nafar, K.: On the associated graphs to a commutative ring. J. Algebra Appl. (2012) doi: 10.1142/S021949881105610.
[28] Beck, I.: Coloring of commutative rings. J. Algebra **116**, 208-226 (1988).
[29] Bollaboás, B.: Graph Theory, An Introductory Course. Springer-Verlag, New York (1979).
[30] Chelvam, T., Asir, T.: Domination in total graph on $\mathbb{Z}_n$. Discrete Math. Algorithms Appl. **3**, 413-421 (2011).
[31] Chelvam, T., Asir, T.: Domination in the total graph of a commutative ring. J. Combin. Math. Combin. Comput. **87**, 147-158 (2013).
[32] Chelvam, T., Asir, T.: Intersection graph of gamma sets in the total graph. Discuss. Math. Graph Theory **32**, 339-354 (2012).
[33] Chelvam, T., Asir, T.: On the Genus of the Total Graph of a Commutative Ring. Comm. Algebra **41**, 142-153 (2013).
[34] Chelvam, T., Asir, T.: On the total graph and its complement of a commutative ring. Comm. Algebra (2013) doi:10.1080/00927872.2012.678956.
[35] Chelvam, T., Asir, T.: The intersection graph of gamma sets in the total graph I. J. Algebra Appl. (2013) doi: 10.1142/S0219498812501988.
[36] Chelvam, T., Asir, T.: The intersection graph of gamma sets in the total graph II. J. Algebra Appl. (2013) doi: 10.1142/S021949881250199X.
[37] Chiang-Hsieh, H.-J., Smith, N. O., Wang, H.-J.: Commutative rings with toroidal zerodivisor graphs. Houston J Math. **36**, 1–31 (2010).
[38] Coykendall, J., Sather-Wagstaff, S., Sheppardson, L., Spiroff, S.: On zero divisor graphs, Progress in Commutative Algebra 2: Closures, finiteness and factorization, edited by (C. Francisco et al. Eds.),Walter Gruyter, Berlin, (2012), 241–299.
[39] DeMeyer, F., DeMeyer, L., Zero divisor graphs of semigroups: J. Algebra. **283**, 190-198 (2005).
[40] DeMeyer, F., McKenzie, T., Schneider, K.: The zero-divisor graph of a commutative semigroup. Semigroup Forum. **65**, 206-214 (2002).
[41] DeMeyer, F., Schneider, K., Automorphisms and zero divisor graphs of commutative rings. In: Commutative rings. Hauppauge, NY: Nova Sci. Publ.; 2002. p. 25–37.
[42] DeMeyer, L., D'Sa, M., Epstein, I., Geiser, A., Smith, K., Semigroups and the zero divisor graph. Bull. Inst. Combin. Appl. **57**, 60-70, (2009).
[43] DeMeyer, L., Greve, L., Sabbaghi, A., Wang, J., The zero-divisor graph associated to a semigroup. Comm. Algebra. **38**, 3370-3391 (2010).
[44] Khashyarmanesh, K., Khorsandi, M.R.: A generalization of the unit and unitary Cayley graphs of a commutative ring. Acta Math. Hungar. **137**, 242–253 (2012).
[45] Maimani, H.R., Pouranki, M.R., Tehranian, A., Yassemi, S.: Graphs attached to rings revisited. Arab. J. Sci. Eng. **36**, 997-1011 (2011).
[46] Maimani, H. R., Pournaki, M. R., Yassemi, S., Zero-divisor graph with respect to an ideal. Comm. Algebra. **34**, 923-929 (2006).
[47] Maimani, H.R., Wickham, C., Yassemi, S.: Rings whose total graphs have genus at most one. Rocky Mountain J. Math. **42**, 1551-1560 (2012).
[48] Mojdeh1, D. A., Rahimi, A. M: Domination sets of some graphs associated to commutative ring. Comm. Algebra **40**, 3389–3396 (2012).
[49] Mulay, S. B.: Cycles and symmetries of zero-divisors. Comm Algebra. **30**, 3533-3558 (2002).
[50] Nikandish, R., Nikmehr, M. J., Bakhtyiari, M.: Coloring of the annihilator graph of a commutative ring, J. Algebra Appl. **15**(07) (2016). DOI: 10.1142/S0219498816501243
[51] Pucanović, Z., Petrović, Z.: On the radius and the relation between the total graph of a commutative ring and its extensions. Publ. Inst. Math.(Beograd)(N.S.) **89**, 1-9 (2011).
[52] Redmond, S. P., An ideal-based zero-divisor graph of a commutative ring. Comm Algebra. **31**, 4425–4443 (2003).
[53] Smith, N. O: Planar zero-divisor graphs, Comm. Algebra **35**, 171-180 (2007).

[54] Sharma, P.K., Bhatwadekar, S.M.: A note on graphical representations of rings. J. Algebra **176**, 124-127 (1995).
[55] Shekarriz, M.H., Shiradareh Haghighi, M.H., Sharif, H.: On the total graph of a finite commutative ring. Comm. Algebra **40**, 2798-2807 (2012).
[56] Visweswaran, S., Patel, H. D.: A graph associated with the set of all nonzero annihilating ideals of a commutative ring, Discrete Math. Algorithm. Appl. **06**, (2014) [22 pages] DOI: 10.1142/S1793830914500475
[57] Wickham, C.: Classification of rings with genus one zero-divisor graphs. Comm Algebra. **36**, 325–345 (2008).

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